Exercises for the lecture course Algebraic Topology II – Sheet 10

University of Bonn, summer term 2025

Exercise 37. Compute the first Chern class of the principal S^1 -bundle over S^2 given by the Hopf fibration.

Exercise 38. The infinite dihedral group D_{∞} is defined by the presentation $\langle s, t | sts = t^{-1}, s^2 = 1 \rangle$.

- (a) Show that there is a fibration $S^1 \to BD_{\infty} \to \mathbb{RP}^{\infty}$;
- (b) Compute $H_n(BD_\infty; \mathbb{Z})$ for $n \in \mathbb{Z}^{\geq 0}$.

Exercise 39. Let M be a closed connected 3-manifold whose fundamental group is perfect, i.e. $\pi = [\pi, \pi]$. Consider a prime p. Let $f: M \to S^3$ be a map of a degree which is prime to p. Consider a pullback

$$\begin{array}{c} \overline{E} \xrightarrow{\overline{f}} E \\ \downarrow & \downarrow^{q} \\ M \xrightarrow{f} S^{3} \end{array}$$

where q is a fibration. Let \mathcal{H}_* be a homology theory with values in \mathbb{F}_p -modules satisfying the disjoint union axiom.

Prove or disprove that $\mathcal{H}_n(\overline{f}): \mathcal{H}_n(\overline{E}) \to \mathcal{H}_n(E)$ is an isomorphism for all $n \in \mathbb{Z}$.

Exercise 40. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of path connected spaces with a CWcomplex B as basis. Suppose that the action of $\pi_1(B)$ on $H_q(F)$ given by the fiber
transport is trivial. Denote by $E_r^{p,q}$ be the Leray-Serre spectral sequence for the fibration p. Prove or disprove:

- (a) The map $H^n(i): H^n(E) \to H^n(F)$ factorizes as the composite $H^n(E) \xrightarrow{\alpha} E_{\infty}^{0,n} \xrightarrow{\beta} H^n(F)$ for an epimorphism α and a monomorphism β ;
- (b) The map $H^n(p): H^n(B) \to H^n(E)$ factorizes as the composite $H^n(B) \xrightarrow{\alpha} E_{\infty}^{n,0} \xrightarrow{\beta} H^n(E)$ for an epimorphism α and a monomorphism β ;

⁰Hand-in Monday 23.06.