Exercises for the lecture course Algebraic Topology II – Sheet 9

University of Bonn, summer term 2025

Exercise 33. Let F be a field and G be a finite group. Prove or disprove that the following assertions are equivalent:

(a) FG is semisimple;

(b) The FG-module F whose underlying F-module is F and on which G acts trivial is projective;

(c) The order $|G|$ of G is invertible in F.

Exercise 34. Let $p\colon E \to B$ be a principal G-bundle for the discrete finite group G. Prove or disprove that $H^n(E;\mathbb{Q})^G$ is isomorphic to $H^n(B;\mathbb{Q})$ for $n \in \mathbb{Z}^{\geq 0}$.

Exercise 35. Let $F \to E \to B$ be a fibration where F and B are path connected closed non-orientable 2-manifolds. Suppose that the fiber transport is trivial. Prove or disprove:

$$H_3(E) \cong_{\mathbb{Z}} \operatorname{Tor}_1^{\mathbb{Z}}(H_1(B), H_1(F)).$$

Exercise 36. Let X be a space. We call a class $a \in H_n(X, \mathbb{Z})$ *realisable* if there exists a map $f\colon M \to X$ with M a closed oriented n-manifold such that $a = f_*[M]$. Prove:

(a) The homomorphisms

$$\Omega_n(X) \to H_n(X, \mathbb{Z}), \quad (f\colon M \to X) \mapsto f_*[M]$$

extend to a transformation of homology theories;

(b) Every class in degrees $n \leq 5$ is realisable;

(c) For every n, every class $a \in H_n(X)$ is rationally realisable, i.e., there exists $N > 0$ such that Na is realisable.

[0]Hand-in Monday 16.06.