Exercises for the lecture course Algebraic Topology II – Sheet 8

University of Bonn, summer term 2025

Exercise 29. Consider $k, l, n \in \mathbb{Z}^{\geq 0}$ for which there exists a fibration $S^k \to S^n \to S^l$. Prove or disprove that then $l = k + 1$ and $n = k + l$ holds.

Exercise 30. Consider fibration of closed connected smooth manifolds $F \to E \to B$. Prove or disprove:

(a) We have $\dim(E) = \dim(F) + \dim(B)$;

(b) If E is orientable, then B and F are orientable;

(c) If B and F are orientable, then E is orientable;

(d) If E is the total space of a principal S^1-bundle $S^1 \to E \to B$, then $\chi(E)$ vanishes.

Exercise 31. Let $\mathcal{H}_*$ be a homology theory satisfying the disjoint union axiom. Consider a pullback of fibrations with CW-complexes as base space

$$\begin{tikzcd} E_0 \arrow[r, "\overline{f}"] \arrow[d, "p_0"'] & E_1 \arrow[d, "p_1"] \\ B_0 \arrow[r, "f"'] & B_1. \end{tikzcd}$$

Suppose that for any $n \in \mathbb{Z}$, any $b \in B_1$, and any loop w in B_1 at b the map $\mathcal{H}_n(p_1^{-1}(b)) \to \mathcal{H}_n(p_1^{-1}(b))$ induced by the element $\tau_1(w_1) \in [p_1^{-1}(b), p_1^{-1}(b)]$ given by the fiber transport is the identity and that $H_n(f)\colon H_n(B_0) \to H_n(B_1)$ is bijective.

Prove or disprove that the map $\mathcal{H}_n(\overline{f})\colon \mathcal{H}_n(E_0) \to \mathcal{H}_n(E_1)$ is bijective for every $n \in \mathbb{Z}$.

Exercise 32. Let $F\colon E \to B$ be a fibration such that the fiber transport is trivial. Let R be a principal ideal domain. Suppose $H_i(F; R)$ and $H_i(B; R)$ are finitely generated for all $i \in \mathbb{Z}^{\geq 0}$ and non-trivial only for finitely many values of i.

(a) Show that $H_i(E; R)$ is finitely generated for all $i \in \mathbb{Z}^{\geq 0}$ and non-trivial only for finitely many values of i;

(b) Show for the Betti numbers, which are define by $b_i(X; R) := \mathrm{rk}_R(H_i(X; R))$,

$$\sum_{i \geq 0} b_i(E; R) \leq \Big(\sum_{j \geq 0} b_j(F; R)\Big) \cdot \Big(\sum_{k \geq 0} b_k(B; R)\Big);$$

(c) Suppose that the inequality above is an equality and R is a field. Prove or disprove

$$H_n(E; R) = \bigoplus_{i \geq 0} H_i(F; R) \otimes_R H_{n-i}(B; R).$$

[0]Hand-in Monday 02.06.