Exercises for the lecture course Algebraic Topology II – Sheet 7

University of Bonn, summer term 2025

Exercise 25. Let A be a $\mathbb{Z}$-module. Let $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ be a nested sequence of $\mathbb{Z}$-submodules of A such that $A = \bigcup_{i \in \mathbb{N}} A_i$ holds.

Prove or disprove that for every $n \in \mathbb{Z}^{\geq 0}$ we get

$$H_n(K(A,1);\mathbb{Z}) = \operatorname{colim}_{i\to\infty} H_n(K(A_i,1);\mathbb{Z}).$$

Exercise 26. Let $f\colon X \to Y$ be a map of CW-complexes. Consider $d \in \mathbb{Z}^{\geq 0}$ and a homology theory $\mathcal{H}_*$ with values in R-modules satisfying the disjoint union axiom such that $\mathcal{H}_i(\{\bullet\}) = \{0\}$ holds for $i \leq -1$. Suppose that $H_i(f;\mathbb{Z})\colon H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$ is bijective for $i < d$ and surjective for $i = d$.

Prove or disprove that $\mathcal{H}_i(f)\colon \mathcal{H}_i(X) \to \mathcal{H}(Y)$ is bijective for $i < d$ and surjective for $i = d$.

Exercise 27. Let X be a finite CW-complex. Let $\mathcal{H}_*$ homology theory with values in $\mathbb{Q}$-modules satisfying the disjoint union axiom such that $\mathcal{H}_i(\{\bullet\}) \neq \{0\}$ holds only for finitely many $i \in \mathbb{Z}$ and $\mathcal{H}_i(\{\bullet\})$ is finitely generated for every $i \in \mathbb{Z}$.

Prove or disprove that $\mathcal{H}_n(X)$ is finitely generated for all $i \in \mathbb{Z}$ and we get for the Euler characteristic

$$\chi(X) \cdot \sum_{n\in\mathbb{Z}} (-1)^n \cdot \dim_{\mathbb{Q}}(\mathcal{H}_n(\{\bullet\})) = \sum_{n\in\mathbb{Z}} (-1)^n \cdot \dim_{\mathbb{Q}}(\mathcal{H}_n(X)).$$

Exercise 28. Let $\mathcal{H}_*$ be any homology theory with values in $\mathbb{Z}$-modules satisfying the disjoint union axiom such that $\mathcal{H}_i(\{\bullet\}) = \{0\}$ holds for $i \leq -1$. Let $f\colon X \to Y$ be a map of connected finite CW-complexes. Suppose that $\mathcal{H}_i(f)\colon \mathcal{H}_i(X) \to \mathcal{H}_i(Y)$ is bijective for all $i \in \mathbb{Z}$.

Prove or disprove that $H_i(f;\mathbb{Z})\colon H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$ is bijective for all $i \in \mathbb{Z}^{\geq 0}$.

[0]Hand-in Monday 26.05.