Exercises for the lecture course Algebraic Topology II – Sheet 6

University of Bonn, summer term 2025

Exercise 21. Let $\mathbb{Z} \rtimes \mathbb{Z}$ be the semidirect product with respect to the group homomorphism $\mathbb{Z} \to \operatorname{aut}(\mathbb{Z})$ sending $m \in \mathbb{Z}$ to the automorphism $(-1)^m \cdot \mathrm{id}_{\mathbb{Z}}$. (Note that $\mathbb{Z} \rtimes \mathbb{Z}$ has the presentation $\langle t, s \mid sts^{-1} = t^{-1} \rangle$.) Let K be the Klein bottle K which is the quotient of $\mathbb{R}^2$ by the free $\mathbb{Z} \rtimes \mathbb{Z}$-action for which t and s act by sending (r_1, r_2) to $(r_1 + 1, r_2)$ and $(-r_1, r_2 + 1)$ respectively.

(a) Show that K is a closed 2-dimensional manifold;

(b) Compute $\pi_1(K)$, $H_n(K;\mathbb{Z})$, $H^n(K;\mathbb{Z})$, $H_n(K;\mathbb{F}_2)$, and $H^n(K;\mathbb{F}_2)$ for $n \geq 0$;

(c) Compute the first Stiefel Whitney class $w_1(M) \in H^1(K;\mathbb{F}_2)$;

(d) Decide whether K is orientable and determine its orientation covering.

Exercise 22. Prove Lemma 2.13 of the script saying that the adjunction homomorphism

$$\mathrm{ad}\colon \hom_{R\mathcal{D}}(M \otimes_{R\mathcal{C}} B, N) \to \hom_{R\mathcal{C}}(M, \hom_{R\mathcal{D}}(B, N))$$

is bijective and natural.

Exercise 23. Let X be a CW-complex such that $H_n(X;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(\{\bullet\};\mathbb{Z})$ holds for $n \in \mathbb{Z}^{\geq 0}$. Prove or disprove that for any homology theory $\mathcal{H}_*$ with values in R-modules satisfying the disjoint union axiom the R-modules $\mathcal{H}_n(X)$ and $\mathcal{H}_n(\{\bullet\})$ are isomorphic for $n \in \mathbb{Z}$.

Exercise 24. Consider $d \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$. Compute $\mathbb{Q} \otimes_{\mathbb{Z}} K_n(\mathbb{RP}^d)$ for $n \in \mathbb{Z}$ for the complex topological K-homology K_*.

[0]Hand-in Monday 19.05.