Exercises for the lecture course Algebraic Topology II – Sheet 4

University of Bonn, summer term 2025

Aufgabe 13. Let X be a CW-complex which is of finite type, i.e., each i-skeleton is finite. Suppose that $\mathcal{H}_*$ is a homology theory with values in $\mathbb{Z}$-modules which satisfies the disjoint union axiom and $\mathcal{H}_m(\{\bullet\}) = 0$ for $m < 0$. Suppose that $\mathcal{H}_m(\{\bullet\})$ is finitely generated for all $m \in \mathbb{Z}^{\geq 0}$.

Prove or disprove that $\mathcal{H}_n(X)$ is finitely generated for every $n \in \mathbb{Z}$ and vanishes for $n < 0$.

Aufgabe 14. Decide whether Ω_* satisfies the weak equivalence axiom: Prove or disprove that for any weak homotopy equivalence $f\colon X \to Y$, the induced map $\Omega_n(f)\colon \Omega_n(X) \to \Omega_n(Y)$ is an isomorphism for all $n \in \mathbb{Z}$.

Aufgabe 15. Prove or disprove:

(a) Let $\mathcal{H}_*$ be homology theory with values in R-modules. Let $f\colon S^1 \to S^1$ be the map sending z to z^d for $d \in \mathbb{Z}$. Then the induced map $\mathcal{H}_n(f)\colon \mathcal{H}_n(S^1, \{1\}) \to \mathcal{H}_n(S^1, \{1\})$ can be identified with the map $\mathcal{H}_{n-1}(\{\bullet\}) \to \mathcal{H}_{n-1}(\{\bullet\})$ given by multiplication with d;

(b) We have:
$$\mathcal{N}_n(\mathbb{RP}^2) \cong_{\mathbb{F}_2} \mathcal{N}_n(\{\bullet\}) \oplus \mathcal{N}_{n-1}(\{\bullet\}) \oplus \mathcal{N}_{n-2}(\{\bullet\}).$$

Aufgabe 16. Let G be a compact Lie group. Show that its tangent bundle is trivial. Describe an (interesting) construction which assigns to G an element in the stable stem π_n^s for $n = \dim(G)$.

[0]Hand-in Monday 05.05.