Exercises for the lecture course Algebraic Topology II – Sheet 2

University of Bonn, summer term 2025

Aufgabe 5. Show that we obtain a transformation of homology theories with values in $\mathbb{F}_2$-modules $T_*\colon \mathcal{N}_* \to H_*(-;\mathbb{F}_2)$ by assigning to an element $[f\colon (M,\partial M) \to (X,A)] \in \mathcal{N}_n(X,A)$ the image of the fundamental class $[M,\partial M] \in H_n(M,\partial M;\mathbb{F}_2)$ under the homomorphism $H_n(M,\partial M;\mathbb{F}_2) \to H_n(X,A;\mathbb{F}_2)$ induced by f. Show that $T_n(X)\colon \mathcal{N}_n(X) \to H_n(X;\mathbb{F}_2)$ is bijective for any CW-complex X and any $n \in \{0,1\}$.

Aufgabe 6.

(a) Give the definition of the signature of a closed oriented smooth manifold and of the Euler characteristic of a closed smooth manifold and list their basic properties (without giving proofs);

(b) Construct using the signature a homomorphisms of graded $\mathbb{Z}$-algebras $s_*\colon \Omega_* \to \mathbb{Z}[x]$ for $|x| = 4$ and $u_*\colon \mathbb{Z}[x] \to \Omega_*$ satisfying $s_* \circ u_* = \mathrm{id}_{\mathbb{Z}[x]}$ for the oriented bordism ring Ω_*;

(c) Construct using the Euler characteristic a homomorphisms of graded $\mathbb{F}_2$-algebras $e_*\colon \mathcal{N}_* \to \mathbb{F}_2[y]$ for $|y| = 2$ and $v_*\colon \mathbb{F}_2[y] \to \mathcal{N}_*$ satisfying $e_* \circ v_* = \mathrm{id}_{\mathbb{F}_2[y]}$ for the unoriented bordism ring $\mathcal{N}_*$.

Aufgabe 7. Compute the topological K-theory $K^*(\mathbb{CP}^d)$ for $d \in \mathbb{Z}^{\geq 1}$ using the facts that K^* is 2-periodic and we have $K^0(\{\bullet\}) \cong \mathbb{Z}$ and $K^1(\{\bullet\}) \cong \{0\}$;

Aufgabe 8. Let M be a closed n-dimensional smooth submanifold of $\mathbb{R}^{n+1}$ for $n \in \mathbb{Z}^{\geq 1}$. Prove or disprove that its normal bundle $\nu(M \subseteq \mathbb{R}^n)$ is trivial if and only if $H_n(M;\mathbb{Z}) \cong \mathbb{Z}^{|\pi_0(M)|}$ holds.

[0] Hand-in Monday 21.04.