Exercises for the lecture course Algebraic Topology II – Sheet 1

University of Bonn, summer term 2025

Aufgabe 1. Let M be a simply connected closed 4-manifold whose Euler characteristic $\chi(M)$ is 2. Prove or disprove that M is homotopy equivalent to S^4.

Aufgabe 2. Let X and Y be CW-complexes and ξ and η vector bundles over X and Y. Prove that there are pointed homeomorphisms

$$\begin{aligned} \operatorname{Th}(\xi \times \eta) &\xrightarrow{\cong} \operatorname{Th}(\xi) \wedge \operatorname{Th}(\eta); \\ \operatorname{Th}(\xi \oplus \underline{\mathbb{R}^k}) &\xrightarrow{\cong} S^k \wedge \operatorname{Th}(\xi). \end{aligned}$$

Aufgabe 3. Compute $\pi_0^s(X)$ for a connected CW-complex X.

Aufgabe 4. Prove:

(a) There exists a self-homomotopy equivalence $f\colon \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ which is not homotopic to the identity.

(b) There exists a fibration $\mathbb{CP}^\infty \to E \to S^1$ such that E is homotopy equivalent to the mapping torus of f;

(c) We have

$$\pi_n(T_f) = \begin{cases} \mathbb{Z} & \text{if } n = 1, 2; \\ \{0\} & \text{otherwise;} \end{cases}$$

(d) The mapping torus T_f is not homotopy equivalent to a product of Eilenberg-MacLane spaces.

[0]Hand-in Monday 14.04.