Algebraic Topology I

The AT1 Tutors

Quick Questions 1

December 23, 2024

Questions

Decide whether the following statements are true or false, answer the question, or do the required task where appropriate. Justify your answers!

1. If X is path-connected and Y is at least 1-connected, then $\pi_2(X \vee Y) \cong \pi_2(X) \oplus \pi_2(Y)$.
2. There are infinitely many homotopy classes of maps $\mathbb{R}\mathrm{P}^3 \to S^3$.
3. There are infinitely many homotopy classes of maps $\mathbb{R}\mathrm{P}^2 \to S^2$.
4. How many principal $\mathbb{Z}/2\mathbb{Z}$-bundles over $\mathbb{R}\mathrm{P}^n$ are there for each $n \geq 1$?
5. Let $F\colon \mathrm{Top}_* \to \mathrm{Set}$ be a functor that preserves finite products. Let G be a topological group. Prove that there exists a preferred group structure on $F((G,e))$.
6. If $A \hookrightarrow X$ is a cofibration, then $X \times \{0,1\} \cup A \times I$ is a retract of $X \times I$.
7. Let $p\colon S^3 \to S^2$ be the Hopf fibration. Is $p^*\colon \pi_2(S^2) = [S^2, S^2]_* \to [S^3, S^2]_* = \pi_3(S^2)$ a group homomorphism?
8. Given two based path-connected CW-complexes (X, x_0) and (Y, y_0), if two maps $f, g\colon (X, x_0) \to (Y, y_0)$ induce the same homomorphism $f_* = g_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ for all $n \geq 1$, then they are homotopic. In particular, if f induces the trivial map on homotopy groups, then f is nullhomotopic.
9. The fundamental group is already stable. In other words, given a path-connected space X, we always have that $\pi_1 X \cong \pi_1^s X$ already.
10. If $f\colon X \to Y$ is a weak homotopy equivalence, then $\Sigma f\colon \Sigma X \to \Sigma Y$ is a weak homotopy equivalence, too.
11. If $p\colon E \to B$ and $p'\colon E' \to B$ are fibrations over the same path-connected base space B such that E is homotopy equivalent to E', then also $F_b \simeq F'_b$

for all $b \in B$ where $F_b := p^{-1}(b)$ and $F'_b := p'^{-1}(b)$ are the respective fibres over the point b.

12. You have learned about classifying spaces BG. Go to the literature and look up the definition of group (co-)homology in terms of BG.