Exercises for the lecture course Algebraic Topology I – Sheet 12

University of Bonn, winter term 24/25

Aufgabe 41. Decide which of the following spaces are Eilenberg-MacLane spaces of type (G, n). If the answer is yes, specify the values for G and n:

(a) S^d for $d \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$;

(b) $\mathbb{RP}^d$ for $d \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$;

(c) $\mathbb{CP}^d$ for $d \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$;

(d) $S^1 \vee S^1$;

(e) T^d for $d \in \mathbb{Z}^{\geq 1}$.

(f) A simply connected 4-manifold.

Aufgabe 42. Let X be an Eilenberg-Mac-Lane space of type (G, n) for $n \geq 2$. Prove or disprove that there is a CW-approximation $K(G, (n-1)) \to \Omega(X, x)$ for every $x \in X$.

Aufgabe 43. (a) Find simply connected pointed spaces X and Y such that the inclusion $X \vee Y \to X \times Y$ is not a weak homotopy equivalence;

(b) Let $\mathbf{E}$ and $\mathbf{F}$ be spectra. Show that we get well-defined spectra $\mathbf{E} \vee \mathbf{F}$ and $\mathbf{E} \times \mathbf{F}$ satisfying $(\mathbf{E} \vee \mathbf{F})_n = E(n) \vee F(n)$ and $(\mathbf{E} \times \mathbf{F})_n = E(n) \times F(n)$ for $n \in \mathbb{Z}$, and that there is an obvious map of spectra $\mathbf{i} \colon \mathbf{E} \vee \mathbf{F} \to \mathbf{E} \times \mathbf{F}$.

Prove or disprove that $\mathbf{i}$ is a weak homotopy equivalence of spectra.

Aufgabe 44. Define the nth homology of a spectrum $\mathbf{E}$ for $n \in \mathbb{Z}$ by

$$H_n(\mathbf{E}) := \operatorname{colim}_{k \to \infty} H_{n+k}(E(k))$$

where the k-th structure map is the composite

$$H_{n+k}(E(k)) \xrightarrow{\sigma_{n+k}(E(k))} H_{n+k+1}(S^1 \wedge E(k)) \xrightarrow{H_{n+k+1}(\text{flip})} H_{n+k+1}(E(k) \wedge S^1) \xrightarrow{H_{n+k+1}(\sigma(k))} H_{n+k+1}(E(k+1)).$$

of the homological suspension isomorphism $\sigma_{n+k}(E(k))$, the map induced by the flip map flip and the homomorphism induced by the structure map $\sigma(k)$.

Decide whether for any abelian group G there is a spectrum $\mathbf{M}(G)$ such that $H_0(\mathbf{M}(G)) \cong G$ holds and $H_n(\mathbf{M}(G))$ vanishes for $n \neq 0$.

[0] Hand-in Monday 13.01.