Exercises for the lecture course Algebraic Topology I – Sheet 11

University of Bonn, winter term 24/25

Aufgabe 41. Let M be a simply connected closed 4-manifold with Euler characteristic $\chi(M) = 2$. Prove or disprove that M is homotopy equivalent to S^4.

Aufgabe 42. Let M be a connected closed orientable 3-manifold such that its fundamental group π is non-trivial and satisfies $\pi = [\pi, \pi]$. (Such a manifold does exist.)

(a) Construct a map $f\colon M \to S^3$ which is not a homotopy equivalence but induces an isomorphism on all homology groups;

(b) Show that $\Sigma f\colon \Sigma M \to \Sigma S^3$ is a homotopy equivalence.

Aufgabe 43. Consider a map $f\colon X \to Y$ of path connected CW-complexes. Decide which of the following assertions are true. (You are allowed to use the previous exercise.)

(a) Suppose that X and Y are simply connected. Then the map f is a homotopy equivalence if and only if its mapping cone is contractible;

(b) Suppose that X and Y are simply connected. Then the map f is a homotopy equivalence if and only if its mapping cone is acyclic, i.e., its homology groups vanish in degree $n \geq 1$;

(c) The map f is a homotopy equivalence if and only if its mapping cone is contractible.

Aufgabe 44. Let X be a path connected CW-complex. Recall that it is aspherical if $\pi_n(X, x)$ vanishes for $n \geq 2$ and any base point x. Show that the following assertions are equivalent:

- X is aspherical;
- $\widetilde{X}$ is contractible;
- $H_i(\widetilde{X})$ is trivial for every $i \geq 2$.

[0] Hand-in Monday 06.01.