Exercises for the lecture course Algebraic Topology I – Sheet 1

University of Bonn, winter term 24/25

Aufgabe 1. Let M be a connected closed 3-manifold whose fundamental group G is perfect, i.e., G agrees with its commutator subgroup $[G, G]$, and non-trivial.

Prove or disprove that there is a map $f\colon M \to S^3$ which is not a homotopy equivalence and induces an isomorphism $H_n(f; A)\colon H_n(M; A) \to H_n(S^3; A)$ for any abelian group A and any $n \geq 0$.

Aufgabe 2. Let $f\colon X \to Y$ be a homotopy equivalence. Show that for any $x \in X$ and $n \geq 1$ the induced map $\pi_n(f, x)\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism.

Aufgabe 3. Compute $\pi_n(T^k \times \mathbb{R}^l, x)$ for all $k, l, n \geq 1$, where T^k is the k-torus.

Aufgabe 4. Consider a path connected space X with base point $x \in X$ and $n \geq 1$.

(a) Let $[S_n] \in H_n(S^n)$ be a generator. Show that we get a well-defined group homomorphism

$$\mathrm{hur}_n\colon \pi_n(X, x) \to H_n(X)$$

by sending $[f]$ represented by the pointed map $f\colon (S^n, s) \to (X, x)$ to the image of the fundamental class $[S^n]$ under the map $H_n(f)\colon H_n(S^n) \to H_n(X)$.

(b) Give for every $n \geq 2$ examples of closed connected orientable manifolds M, N of dimension n such that $\mathrm{hur}_n\colon \pi_n(M, x) \to H_n(M)$ is surjective and $\mathrm{hur}_n\colon \pi_n(N, x) \to H_n(N)$ is trivial.

[0]Hand-in Monday 14.10.